Sydney Girls High School
2023
Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 90 marks** • Attempt Questions 11-16 • Allow about 2 hours and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2023 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

Questions **Marks**

1 Let $z = 1 + \sqrt{3}i$. What is z in mod-arg form? **1**

A. $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

B. $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

C. $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

D. $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

2 Given that $z = 1 + 2i$ is a root of the equation $z^2 - (3+i)z + k = 0$, what is the value of k? **1**

A. $k = 3i$

B. $k = 1 - 2i$

C. $k = 2 - i$

D. $k = 4 + 3i$

3 Given the statement **1**

$$\text{In } \triangle ABC,\ \sin A = \frac{\sqrt{3}}{2} \implies A = 60^\circ.$$

Which of the following is correct?

A. The original statement is false and the converse statement is false.

B. The original statement is false and the converse statement is true.

C. The original statement is true and the converse statement is false.

D. The original statement is true and the converse statement is true.

4 Which of the following expressions is equal to $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$? **1**

A. $\dfrac{1}{\ln x} + C$

B. $\dfrac{1}{(\ln x)^3} + C$

C. $\ln\left(\dfrac{1}{x}\right) + C$

D. $-\dfrac{1}{\ln x} + C$

5 The points A, B and C are collinear where $\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, and $\overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$. **1**

What are the values of a and b?

A. $a = -3,\ b = -2$

B. $a = 3,\ b = -2$

C. $a = -3,\ b = 2$

D. $a = 3,\ b = 2$

6 A particle is moving on a line with simple harmonic motion. At time t seconds it has displacement x metres from a fixed point on the line and velocity v m/s given by **1**

$$v^2 = -\frac{1}{2}x^2 + 2x + \frac{5}{2}.$$

What is the period of the motion?

A. π seconds

B. $\pi\sqrt{2}$ seconds

C. 2π seconds

D. $2\pi\sqrt{2}$ seconds

7 Let z be a complex number where $0 < \operatorname{Arg}(z) < \dfrac{\pi}{4}$. Which of the following is correct? **1**

A. iz lies in the second quadrant and $z - iz$ lies in the first quadrant.

B. iz lies in the second quadrant and $z - iz$ lies in the fourth quadrant.

C. iz lies in the fourth quadrant and $z - iz$ lies in the first quadrant.

D. iz lies in the fourth quadrant and $z - iz$ lies in the second quadrant.

8 The equation $z^5 = 1$ has roots $1, \omega, \omega^2, \omega^3$, and ω^4, where $\omega = e^{i\frac{2\pi}{5}}$. **1**

What is the value of $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$?

A. -5

B. -4

C. 4

D. 5

9 Recall that the probability density function of the standard normal distribution is given by **1**

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad \text{for } -\infty < z < \infty$$

and hence, by the empirical rule,

$$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \approx 0.95.$$

Which of the following integrals has the largest value?

A. $\displaystyle\int_0^{\pi} \cos^5 x\, dx$

B. $\displaystyle\int_{-2}^{2} e^{-\frac{1}{2}x^2}\, dx$

C. $\displaystyle\int_{-\sqrt{3}}^{1} \tan^{-1} x\, dx$

D. $\displaystyle\int_1^{e} \sqrt{\ln x}\, dx$

10 Let a, b and c be positive real numbers. Which of the following expressions has the smallest minimum value? **1**

A. $\dfrac{(a+b)(b+c)(a+c)}{abc}$

B. $\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{a+c}{b}$

C. $\left(a+\dfrac{1}{a}\right)\left(b+\dfrac{1}{b}\right)\left(c+\dfrac{1}{c}\right)$

D. $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

Examination continues overleaf. . .

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question on the writing paper supplied. Start each question on a NEW page. Extra writing paper are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start on a NEW page **Marks**

(a) The complex numbers $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $w = 2e^{i\frac{\pi}{3}}$ are given.

(i) Express z in exponential form. **1**

(ii) Find the value of zw, giving the answer in the form $re^{i\theta}$. **2**

(b) Consider the vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + p\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{c} = -2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$. **3**

For what values of p are $\underset{\sim}{b} + \underset{\sim}{a}$ and $\underset{\sim}{b} - \underset{\sim}{c}$ perpendicular?

(c) Use integration by parts to find $\displaystyle\int xe^x\,dx$. **3**

(d) A particle starts from rest at the origin with acceleration given by **3**

$$a = v^3 + v,$$

where v is the velocity of the particle.

Find an expression for x, the displacement of the particle, in terms of v.

(e) Fully simplify $(i\,\overline{z})^{2023}$, where $z = \cos\frac{\pi}{289} - i\sin\frac{\pi}{289}$. **3**

Examination continues overleaf...

Question 12 (15 marks) Start on a NEW page **Marks**

(a) Prove by contraposition that if $n^3 - n$ is not divisible by 4, then n must be even. **3**

(b) Using a trigonometric substitution, or otherwise, find $\displaystyle\int \frac{1}{\sqrt{(1-x^2)^3}}\,dx$. **4**

Give your answer without trigonometric functions.

(c) Consider the lines $\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$. **3**

Assuming these lines are neither parallel nor perpendicular, determine whether the lines intersect or are skew.

(d) Sketch the region of the complex plane defined by $|z - 3i| < 2|z|$. **3**

(e) A particle undergoes simple harmonic motion with period T seconds and amplitude A cm. **2**

What is its maximum speed?

Examination continues overleaf. . .

Question 13 (15 marks) Start on a NEW page **Marks**

(a) (i) Given that **3**

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2},$$

find the values of A, B, and C.

(ii) Hence, or otherwise, find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{5\cos x}{1+2\sin x+\cos x}\,dx$ **3**

using the substitution $t = \tan\dfrac{x}{2}$.

(b) Prove that $\sqrt[3]{p}$ is irrational, where p is a prime number. **3**

(c) Use mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

(d) Let x and y be real numbers.

(i) Give a counterexample to disprove that $2xy \geq xy$. **1**

(ii) Prove that $x^2 + y^2 \geq xy$. **2**

Examination continues overleaf. . .

Question 14 (15 marks) Start on a NEW page **Marks**

(a) The displacement, x metres, of a particle P from the origin O at time t seconds is given by

$$x = 6\cos\left(2t + \frac{\pi}{4}\right) + \cos(2t).$$

(i) Show that P is moving in simple harmonic motion about O. **3**

(ii) Find the amplitude of this motion, correct to 1 decimal place. **3**

(b) Consider a sphere $\mathcal{S}$, centred at point $C(2, -1, 0)$ with radius $\sqrt{29}$.

Consider also the line ℓ with parametric equations

$$x = \lambda + 1,\ \ y = \lambda,\ \ z = 2\lambda + 3.$$

(i) Find the vector equation of line ℓ, writing your answer in the form $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{d}$, where $\underset{\sim}{a}$ and $\underset{\sim}{d}$ are expressed as column vectors. **1**

It is known that ℓ intersects the surface of $\mathcal{S}$ at points P and Q.

(ii) Find the coordinates of P and Q. **3**

(iii) Hence, or otherwise, determine whether PQ is a diameter of $\mathcal{S}$, showing all necessary working. **1**

(c) In the Argand diagram, points A, B, C and D represent the complex numbers α, β, γ and δ respectively.

(i) If $\alpha + \gamma = \beta + \delta$, show that $ABCD$ is a parallelogram. **2**

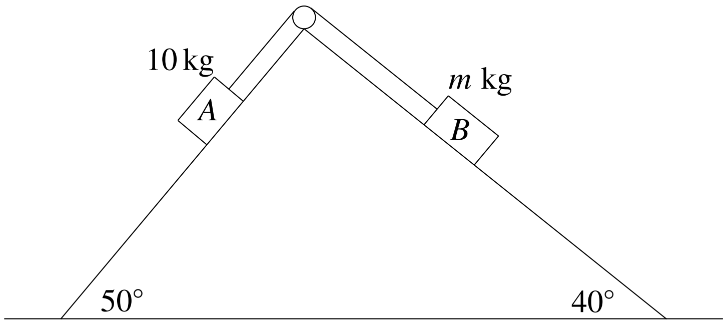
(ii) If $ABCD$ is a square with vertices in anticlockwise order, show that **2**

$$\gamma + i\alpha = \beta + i\beta.$$

Examination continues overleaf. . .

Question 15 (15 marks) Start on a NEW page **Marks**

(a) Two bodies, A and B, are attached by a light, inextensible string. The string is placed over a smooth pulley on the ridge as shown.



The body A has a mass of 10 kg and is supported against a smooth plane of angle 50°. The body B has a mass of m kg and is supported against a smooth plane of angle 40°.

The two bodies are at rest before being released. After they are released, A moves up the plane and B moves down the plane at a constant velocity.

(i) Briefly explain why the net force in the direction of motion for each body is zero. In your explanation, you must make reference to the given velocity. **1**

(ii) By considering the forces acting on each body, or otherwise, determine the value of m, correct to 1 decimal place. **3**

(b) You are given that set of rational numbers $\mathbb{Q}$ is *closed* under the four operations.

That is, if $r, s \in \mathbb{Q}$, then

- $r + s \in \mathbb{Q}$
- $r - s \in \mathbb{Q}$
- $rs \in \mathbb{Q}$
- $\frac{r}{s} \in \mathbb{Q}$

(Do NOT prove this.)

Suppose ABC is a triangle such that each side length is a rational number.

Let $a = BC$, $b = AC$, $c = AB$ and $\alpha = \angle BAC$.

(i) By using the cosine rule in triangle ABC, or otherwise, show that $\cos\alpha$ is rational. **1**

(ii) Using de Moivre's theorem and the binomial expansion of $(\cos\alpha + i\sin\alpha)^5$, or otherwise, deduce that $\cos 5\alpha$ is rational. **3**

Examination continues overleaf...

(c) Suppose that line ℓ_1 has vector equation

$$\underset{\sim}{r} = \lambda \begin{pmatrix} \cos\phi + \sqrt{3} \\ \sqrt{2}\sin\phi \\ \cos\phi - \sqrt{3} \end{pmatrix}$$

and that line ℓ_2 has vector equation

$$\underset{\sim}{r} = \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

where $\lambda,\ \mu \in \mathbb{R}$.

(i) Show that the acute angle θ between ℓ_1 and ℓ_2 is independent of ϕ. **3**

(ii) A plane has equation $x - z = 4\sqrt{3}$. The line ℓ_2 meets this plane at C. **1**

Find the coordinates of C.

(iii) The line ℓ_1 intersects the plane $x - z = 4\sqrt{3}$ at the point P. **3**

Show that as ϕ varies, P describes a circle of centre C and radius $2\sqrt{2}$.

Examination continues overleaf...

Question 16 (15 marks) Start on a NEW page **Marks**

(a) Let $f(x)$ be a concave down function on a given interval and let x_1, x_2, and x_3 lie in the given interval.
Jensen's inequality states that

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \leq f\left(\frac{x_1 + x_2 + x_3}{3}\right).$$

(Do NOT prove this.)

(i) Show algebraically that $f(x) = \sin x$ is concave down for $0 < x < \pi$. **1**

(ii) Suppose that A, B and C are the angles of a triangle. **2**

By using part (i), or otherwise, show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

Examination continues overleaf...

(b) (i) Show that $\dfrac{x^{2n-1}}{\sqrt{1-x^2}} - \dfrac{x^{2n+1}}{\sqrt{1-x^2}} = x^{2n-1}\sqrt{1-x^2}$. **1**

(ii) For every integer $n \geq 1$, let $I_{2n-1} = \displaystyle\int_0^1 \frac{x^{2n-1}}{\sqrt{1-x^2}}\,dx$. **3**

Using integration by parts and the result from part (i), or otherwise, show that for $n \geq 1$,

$$I_{2n+1} = \left(\frac{2n}{2n+1}\right) I_{2n-1}.$$

(iii) Using part (ii), or otherwise, show that **2**

$$I_{2n+1} = \frac{2^n \times n!}{1 \times 3 \times 5 \times \cdots \times (2n+1)}.$$

(iv) Using part (iii), or otherwise, show that **2**

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \int_0^1 \left[\sum_{n=1}^{\infty}\left(C_n \frac{x^{2n+1}}{\sqrt{1-x^2}}\right)\right] dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

where $C_n = \dfrac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{(2n+1)2^n\, n!}$.

You are given that

$$\int_0^1 \left[\sum_{n=1}^{\infty}\left(C_n \frac{x^{2n+1}}{\sqrt{1-x^2}}\right)\right] dx = \sum_{n=1}^{\infty} C_n \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx.$$

(Do NOT prove this.)

Examination continues overleaf...

(b) (v) The inverse sine function $\sin^{-1} x$ can be defined by the following series: **2**

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} C_n x^{2n+1}$$

where $C_n = \dfrac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{(2n+1)2^n\, n!}$.

Using this definition of $\sin^{-1} x$ and the result from part (iv), or otherwise, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

(vi) Hence, or otherwise, find the limiting value of S if **2**

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

End of paper

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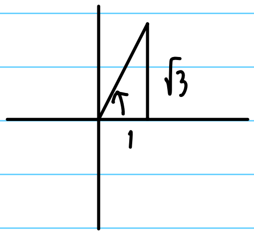
MCQ Solutions

1) $z = 1 + \sqrt{3}i$



$|z| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\arg(z) = \frac{\pi}{3}$

$\therefore z = 2\,\text{cis}\,\frac{\pi}{3}$ $\quad\therefore$ C

2) $P(1+2i) = 0$

$(1+2i)^2 - (3+i)(1+2i) + k = 0$

$1 + 4i - 4 - (3 + 7i - 2) + k = 0$

$-3 + 4i - 1 - 7i + k = 0$

$\therefore k = 4 + 3i$ $\quad\therefore$ D

Note: The conjugate root theorem can not be applied in this question.

3) Original is false ($A = 60^\circ$ is not the only solution)

Converse:

In ΔABC, $A = 60^\circ \Rightarrow \sin A = \frac{\sqrt{3}}{2}$

True $\left(\sin 60^\circ = \frac{\sqrt{3}}{2}\right)$

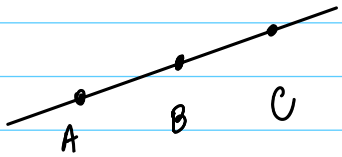
$\therefore$ B

4) $$\int \frac{1}{x(\ln x)^2}\,dx$$

$$= \int \frac{1}{x}(\ln x)^{-2}\,dx$$

$$= \frac{(\ln x)^{-1}}{-1} + C \quad \text{(reverse chain rule)}$$

$$= -\frac{1}{\ln x} + C \qquad \therefore \text{D}$$

5) 

$$\vec{AB} = \lambda \vec{BC}$$

$$2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} - \underset{\sim}{i} - \underset{\sim}{j} = \lambda\left(3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k} - 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} = \lambda\left(\underset{\sim}{i} + (a+1)\underset{\sim}{j} + (b-1)\underset{\sim}{k}\right)$$

$$\begin{cases} \lambda = 1 \\ \lambda(a+1) = -2 \\ \lambda(b-1) = 1 \end{cases}$$

$$\therefore \begin{cases} a+1 = -2 \\ b-1 = 1 \end{cases} \qquad \therefore \begin{cases} a = -3 \\ b = 2 \end{cases} \qquad \therefore \text{C}$$

6) $$\frac{1}{2}v^2 = -\frac{1}{4}x^2 + x + \frac{5}{4}$$

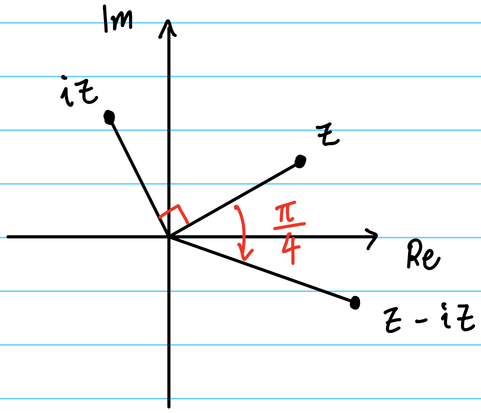
$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}x + 1$$

$$= -\frac{1}{2}(x-2)$$

$$\therefore n = \frac{1}{\sqrt{2}}$$

$$\therefore \text{period} = \frac{2\pi}{\frac{1}{\sqrt{2}}} = 2\pi\sqrt{2} \qquad \therefore \text{D}$$

7)



$$z - iz = z(1-i)$$

$$= z \cdot \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

rotate z by $\frac{\pi}{4}$ clockwise (and enlarge by a factor of $\sqrt{2}$)

$\therefore$ B

8) $z^5 = 1$

$z^5 - 1 = 0$

Factorise $z^5 - 1$ in two different ways:

$$z^5 - 1 = \begin{cases} (z-1)(1+z+z^2+z^3+z^4) \\ (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4) \end{cases}$$

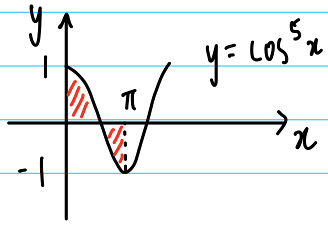
$$\therefore 1 + z + z^2 + z^3 + z^4 = (z-w)(z-w^2)(z-w^3)(z-w^4)$$

let $z = 1$:

$$5 = (1-w)(1-w^2)(1-w^3)(1-w^4)$$

$\therefore$ D

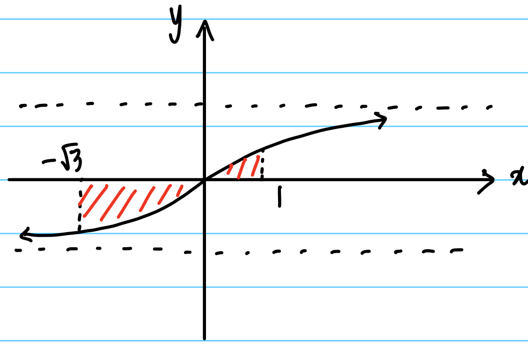
9) • $\int_0^{\pi} \cos^5 x \, dx = 0$



• $\int_{-2}^{2} e^{-\frac{1}{2}x^2} dx \fallingdotseq 0.95 \times \sqrt{2\pi}$

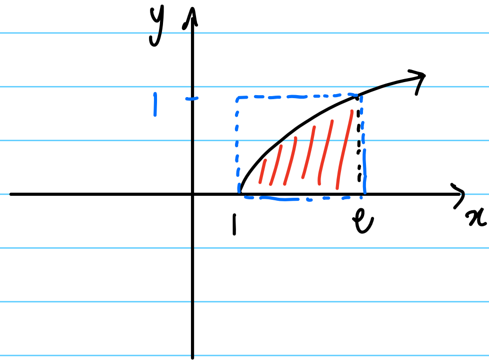
$= 2.38$

• $\int_{-\sqrt{3}}^{1} \tan^{-1} x \, dx < 0$



• $\int_{1}^{e} \sqrt{\ln x} \, dx < (e-1) \times 1$

$= 1.71 \cdots$



$\therefore$ B

10) • $\dfrac{(a+b)(b+c)(a+c)}{abc} \geq \dfrac{2\sqrt{ab}\ 2\sqrt{bc}\ 2\sqrt{ac}}{abc}$

$= \dfrac{8\sqrt{a^2b^2c^2}}{abc}$

$= 8$

• $\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{a+c}{b}$

$= \left(\dfrac{a}{c} + \dfrac{c}{a}\right) + \left(\dfrac{b}{c} + \dfrac{c}{b}\right) + \left(\dfrac{b}{a} + \dfrac{a}{b}\right)$

$\geq 2 + 2 + 2$

$= 6$

• $\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) \geq 2 \times 2 \times 2$

$= 8$

• $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3\sqrt[3]{abc} \cdot 3\sqrt[3]{\frac{1}{abc}}$

$= 9$

$\therefore$ B

Question 11

(a)

(i) $z = 3e^{i\frac{\pi}{4}}$

(ii) $zw = 3e^{i\frac{\pi}{4}} \times 2e^{i\frac{\pi}{3}}$

$= 6e^{i\frac{7\pi}{12}}$

(b) $\underset{\sim}{b} + \underset{\sim}{a} = 3\underset{\sim}{i} + (p+2)\underset{\sim}{j} + 7\underset{\sim}{k}$

$\underset{\sim}{b} - \underset{\sim}{c} = 4\underset{\sim}{i} + (p-4)\underset{\sim}{j} - \underset{\sim}{k}$

$(\underset{\sim}{b} + \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$

$12 + (p+2)(p-4) - 7 = 0$

$p^2 - 2p - 3 = 0$

$(p-3)(p+1) = 0$

$\therefore p = 3, -1$

(c) $\int xe^x \, dx$

$u = x \qquad v' = e^x$

$u' = 1 \qquad v = e^x$

$= xe^x - \int e^x \, dx$

$= xe^x - e^x + C$

Comment

A few students used overcomplicated methods to find $\int e^x dx$.

(d) $v\dfrac{dv}{dx} = v^3 + v$

$$\frac{v\,dv}{v^3+v} = dx$$

$$\int_0^v \frac{dv}{v^2+1} = \int_0^x dx$$

$$\left[\tan^{-1}v\right]_0^v = \left[x\right]_0^x$$

$$x = \tan^{-1}v$$

Comment

Students who converted 'a' into $\dfrac{v\,dv}{dx}$ were generally successful.

A handful of students who integrated without limits had forgotten to evaluate 'C'.

(e) $(i\bar{z})^{2023} = i^{2023}\left(\operatorname{cis}\frac{\pi}{289}\right)^{2023}$

$= (i^4)^{505}\, i^3 \operatorname{cis} 7\pi$ ✓

$= -i \times -1$ ✓

$= i$ ✓

Comment

Some students solved this question using arduous methods.

Some also did not simplify fully, and/or had transcript errors (e.g. writing z as $\cos\frac{\pi}{289} + i\sin\frac{\pi}{289}$)

Question 12

3 (a) Suffices to show that if n is odd, then $n^3 - n$ is divisible by 4. ✓

Let $n = 2k-1$, $k \in \mathbb{Z}$. ✓

$$\therefore n^3 - n = n(n^2-1)$$
$$= n(n-1)(n+1)$$
$$= (2k-1)(2k-2)(2k)$$
$$= 4(2k-1)(k-1)k$$

which is divisible by 4. ✓

4 (b) Let $x = \sin\theta$ ✓

$$dx = \cos\theta \, d\theta$$

$$\therefore \int \frac{dx}{\sqrt{(1-x^2)^3}}$$

$$= \int \frac{\cos\theta \, d\theta}{\sqrt{(1-\sin^2\theta)^3}} \quad ✓$$

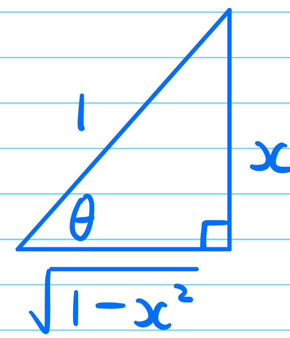
$$= \int \frac{\cos\theta}{(\cos^{\cancel{2}}\theta)^{3/\cancel{2}}}\, d\theta$$

$$= \int \frac{\cancel{\cos\theta}}{\cos^{\cancel{3}\,2}\theta}\, d\theta$$

$$= \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$



(c) Lines intersect if :

3

$$1 + \lambda = 1 - \mu$$

$$\lambda = -\mu \quad (1)$$

$$3 - 4\lambda = 2 + 3\mu$$

$$1 - 4\lambda = 3\mu \quad (2)$$

$$-2 + 7\lambda = -1 + \mu$$

$$-1 + 7\lambda = \mu \quad (3)$$

Sub ① into ②:

$$1 + 4\mu = 3\mu$$
$$\mu = -1$$
$$\therefore \lambda = 1$$

Sub these into ③:

$$-1 + 7(1) = -1$$

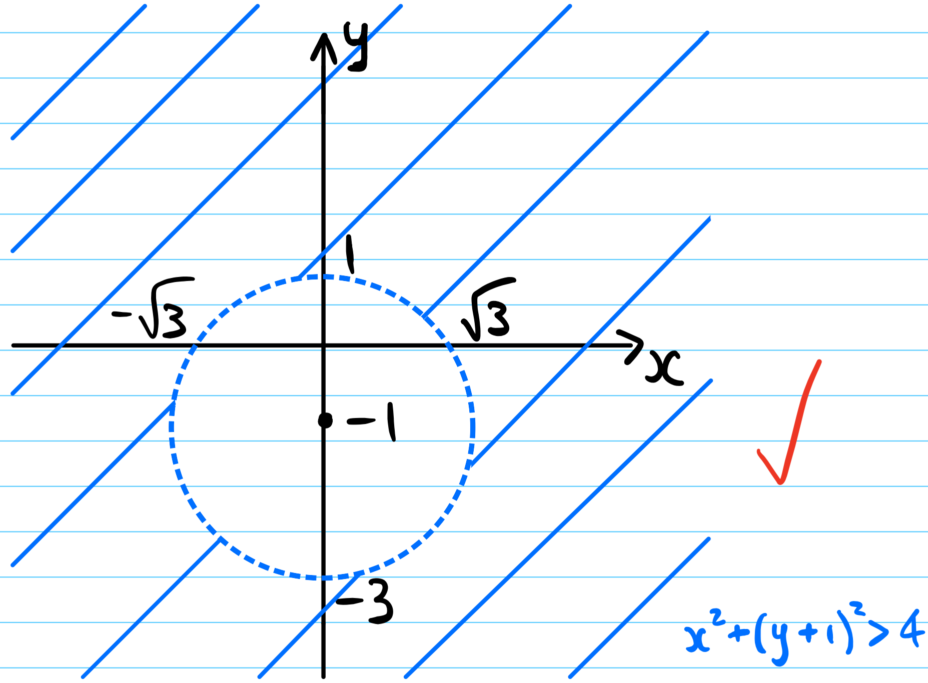
This is <u>false</u>.

$\therefore$ No points of intersection.

(i.e. the lines are skew)

3 (d) Let $z = x + iy$.

$$|x + iy - 3i| < 2|x + iy|$$
$$\sqrt{x^2 + (y-3)^2} < 2\sqrt{x^2 + y^2}$$
$$x^2 + (y-3)^2 < 4x^2 + 4y^2$$
$$3x^2 + 4y^2 - y^2 + 6y - 9 > 0$$
$$3x^2 + 3y^2 + 6y - 9 > 0$$
$$x^2 + y^2 + 2y - 3 > 0$$
$$x^2 + (y+1)^2 > 4$$



2 (e) $T = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{T}$

Let $x = A\cos(nt+\alpha)+c$

$\dot{x} = -An\sin(nt+\alpha)$

$= -\frac{2\pi A}{T}\sin(nt+\alpha)$

$\therefore$ max speed $= \left|-\frac{2\pi A}{T}\right|$

$= \frac{2\pi A}{T}$

Q12 Comments

(a) Very well done. Most students were able to correctly state and prove the contrapositive.

(b) Quite well done. Common errors included using the wrong substitution (e.g. $x = \tan\theta$), algebraic errors, and not expressing the answer in terms of x.

(c) Very well done. There were a variety of correct responses depending on which equations were solved simultaneously first.

(d) Students struggled with this part. The biggest problem was that some students did not know how to start (let $z = x+iy$)

which made it impossible to make progress.

(e) Quite well done. Common errors included negative maximum speed, or expressing the answer in terms of n instead of T.

Question 13 a) i) ③ Marks

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$$

$$5-5x^2 = A(1+x^2) + (Bx+C)(1+2x)$$
$$= A + Ax^2 + Bx + 2Bx^2 + C + 2Cx$$
$$\therefore 5-5x^2 \equiv (A+2B)x^2 + (B+2C)x + (A+C)$$ ✓

Equating Coefficients (or by substitution)

$A+C=5 \quad \ldots ①$

$A+2B=-5 \quad \ldots ②$

$B+2C=0 \quad \ldots ③$ ✓

Solving simultaneously:

③ → ②: $A-4C=-5$

$A = 4C-5 \quad \ldots ④$

④ → ①: $4C-5+C=5$

$5C=10$

$\underline{\underline{C=2}}$

$B=-2(2)$

$\underline{\underline{B=-4}}$

$A=4(2)-5$

$\underline{\underline{A=3}}$.

* Any error made:
  - was deducted one mark.
  - error carried for part ii).

$\therefore A=3,\ B=-4,\ C=2$ ✓

hence:

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{3}{(1+2x)} + \frac{-4x+2}{(1+x^2)}$$

Question 13 a) ii) 3 Marks

Let $I = \int_0^{\pi/2} \frac{5\cos x \, dx}{1+2\sin x+\cos x}$ using $t=\tan\frac{x}{2}$

hence:

$$I = \int_{\tan 0}^{\tan\frac{\pi}{4}} \frac{5\left(\frac{1-t^2}{1+t^2}\right)}{1+2\left(\frac{2t}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{(1+t^2)}$$

$t=\tan\frac{x}{2}$

$\frac{x}{2}=\tan^{-1}t$

$x=2\tan^{-1}t$.

$\frac{dx}{dt}=\frac{2}{1+t^2}$ ✓

$\therefore dx=\frac{2dt}{1+t^2}$

$$= \int_0^1 \frac{10\left(\frac{1-t^2}{1+t^2}\right)dt}{\frac{(1+\cancel{t^2}+4t+1-\cancel{t^2})(\cancel{1+t^2})}{(\cancel{1+t^2})}}$$

* Students should derive $\frac{dx}{dt}$ at all times.

$$= \int_0^1 \frac{10(1-t^2)\,dt}{(1+t^2)(2+4t)}$$

* One mark was deducted for small numerical errors.

$$= \int_0^1 \frac{5-5t^2 \; dt}{(1+2t)(1+t^2)}$$

✓ $$\Rightarrow \int_0^1 \left(\frac{3}{(1+2t)} + \frac{-4t+2}{(1+t^2)}\right)dt$$ from part i)

$$= \int_0^1 \frac{3}{2}\cdot\frac{2}{(1+2t)}dt - \int_0^1 2\times\frac{2t}{(1+t^2)}dt + \int_0^1 \frac{2}{(1+t^2)}dt$$

$$= \frac{3}{2}\Big[\ln|1+2t|\Big]_0^1 - 2\Big[\ln|1+t^2|\Big]_0^1 + \Big[2\tan^{-1}(t)\Big]_0^1$$

$$= \left(\frac{3}{2}\ln 3 - 0\right) - \left(2\ln 2 - 0\right) + \left(2\tan^{-1}(1) - 2\tan^{-1}(0)\right)$$

$$= \frac{3}{2}\ln 3 - 2\ln 2 + 2\times\frac{\pi}{4}$$

✓ $$= \left(\frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}.\right)$$ ← This is an exact answer. No need to go further.

Question 13 b) ③ Marks

Prove that $\sqrt[3]{p}$ is irrational, where p is a prime number.

** This should be proven the same way as showing that $\sqrt{2}$ is irrational, where '2' is a prime number **

* Many students missed the point of the proof arguing that primes have only 2 factors.

Proof by contradiction:

Assume $\sqrt[3]{p}$ is rational where p is prime.

That is:

$\sqrt[3]{p} = \frac{a}{b}$, where a, b are non-zero integers and a and b have no common factors.

$\therefore \; p = \frac{a^3}{b^3}$

That is: $a^3 = p \times b^3$ ✓ ... ①

Since a^3 is divisible by p (a prime) then a must be divisible by p and we can write $a = k \times p$, for integer k. *

From ① $(K \times p)^3 = p \times b^3$

$$K^3 \times p^3 = p \times b^3$$

$$\therefore b^3 = p^2 \times K^3$$

* since b^3 is divisible by p^2
then b must be divisible by p. ✓

$\therefore$ a and b have p as a common prime factor.

However, this contradicts the assumption that a and b have no common factors.

* $\therefore$ By contradiction, the assumption that $\sqrt[3]{p}$ is rational is proven to be false.

Consequently, $\sqrt[3]{p}$ must be rational. (*)

(*) A CLEAR assumption and CLEAR reasoning (*) based on the assumption, was awarded the third (✓) mark.

Question 13 c) ③ Marks

For all integers $n \geq 2$ show that:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Proof by induction:

* Many students did not prove the case for $n=2$ correctly!

For $n=2$:

$LHS = \frac{1}{1^2} + \frac{1}{2^2}$ $= \frac{5}{4}$ $= 1.25$

$RHS = 2 - \frac{1}{2}$ $= \frac{3}{2}$ $= 1.5$

Since $1.25 < 1.5$ ($LHS < RHS$)

then statement is true for $n=2$.

Assume statement is true for $n=k$, where k is an integer, $k \geq 2$.

That is:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$$ ✓

hence

$$2 > \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}\right) + \frac{1}{k} \quad (*)$$

Prove true for $n=k+1$. That is show that:

$$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{(k+1)}$$

or equivalently, show that:

$$2-\frac{1}{(k+1)}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\right) > 0$$

$$\text{LHS} = 2-\frac{1}{(k+1)}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\right)$$

$$> \underbrace{\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{k^2}\right)+\frac{1}{k}}_{\text{from assumption}(*)}-\frac{1}{(k+1)}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}\right)-\frac{1}{(k+1)^2}$$

$$= \frac{1}{k}-\frac{1}{(k+1)}-\frac{1}{(k+1)^2}$$

$$= \frac{(k+1)^2-k(k+1)-k}{k(k+1)^2}$$

$$= \frac{k^2+2k+1-k^2-k-k}{k(k+1)^2}$$

$$= \frac{1}{k(k+1)^2}$$

$$> 0, \text{ since } k \geq 2$$

* Completing the proof with this method, avoided many algebraic errors that were made.

* Justification for expressions being either greater than zero or less than zero needed to be made.

Hence the statement is proven for $n=2$ and $n=k+1$, when it is true for $n=k$.

By the Principle of Mathematical Induction the statement is true for all integers $n \geq 2$.

Question 13 d) ① + ② Marks.

(i) Disprove $2xy \geqslant xy$ by counterexample.

Let $x = -1$, $y = 1$

$2xy = 2 \times -1 \times 1$
$= -2$

$xy = -1 \times 1$
$= -1$

$\therefore$ Statement $2xy \geqslant xy$ does not hold true since $-2 \geqslant -1$ is false, $-2 < -1$.

* Part i) alerts students to 2 possible cases for part ii)

(ii) Consider 2 cases for the values of x and y.

Case 1: x and y such that $xy \geqslant 0$.

If $xy \geqslant 0$ then $xy + xy \geqslant 0 + xy$

$2xy \geqslant xy \quad \ldots ①$

Now $(x-y)^2 \geqslant 0$

$x^2 - 2xy + y^2 \geqslant 0$

$x^2 + y^2 \geqslant 2xy$

$\therefore x^2 + y^2 \geqslant xy$. (since $2xy \geqslant xy$ from ①)

* Students hastily completed this part without annotating their work: $xy \geqslant 0$.

Case 2 : x and y such that $xy < 0$.

(That is: $0 > xy$)

$x^2 \geqslant 0 \qquad y^2 \geqslant 0$

$x^2 + y^2 \geqslant 0$

$\therefore \; x^2 + y^2 > xy$, since $0 > xy$.

Hence in either case: $x^2 + y^2 \geqslant xy$.

* Most students ignored the possibility of the case $xy < 0$.

Question 14 a) i) ③ Marks

$$x = 6\cos\left(2t+\frac{\pi}{4}\right) + \cos(2t)$$

$$\dot{x} = -6(2)\sin\left(2t+\frac{\pi}{4}\right) + (-2)\sin(2t)$$
$$= -12\sin\left(2t+\frac{\pi}{4}\right) - 2\sin(2t) \quad ✓$$

$$\ddot{x} = (-12)(2)\cos\left(2t+\frac{\pi}{4}\right) - 2(2)\cos(2t)$$
$$= -24\cos\left(2t+\frac{\pi}{4}\right) - 4\cos(2t)$$
$$= -4\left(6\cos\left(2t+\frac{\pi}{4}\right) + \cos(2t)\right)$$

$\therefore \ddot{x} = -4x$ ✓

$\therefore \ddot{x} = -n^2 x$

Since $\ddot{x} = -n^2(x-c)$ where $n=2$ and $c=0$, P is moving in simple harmonic motion about O with period $\frac{2\pi}{2} = \pi$.

✓

* Students need to write a statement to justify why they know that the particle is moving in SHM.

* A reference to $\ddot{x} = -n^2 x$ had to be made at the very least.

Question 14 a) ii) ③ Marks

$$x = 6\cos\left(2t + \frac{\pi}{4}\right) + \cos(2t)$$
$$= 6\left[\cos(2t)\cos\frac{\pi}{4} - \sin(2t)\sin\frac{\pi}{4}\right] + \cos(2t)$$
$$= \frac{6\sqrt{2}}{2}\left[\cos(2t) - \sin(2t)\right] + \cos(2t)$$
$$x = (3\sqrt{2}+1)\cos(2t) - 3\sqrt{2}\sin(2t)$$ ✓

By Auxilliary angle method:

$$x \equiv R\cos(2t+\alpha)$$
$$x \equiv R\cos 2t\cos\alpha - R\sin 2t\sin\alpha$$

$$R\cos\alpha = 3\sqrt{2}+1$$
$$R\sin\alpha = 3\sqrt{2}$$ ✓

$$R^2 = (3\sqrt{2}+1)^2 + (3\sqrt{2})^2$$
$$R = \sqrt{6\sqrt{2}+37}$$
$$R \doteqdot 6.744$$

$\therefore$ amplitude $= 6.7$ (1 dec. place). ✓

* Alternate methods were considered but these were more prone to errors.

Question 14 b) ① + ③ + ①

i) $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda+1 \\ \lambda \\ 2\lambda+3 \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ ✓

ii) Equation of sphere:

$\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| = \sqrt{29}$

$\left| \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| = \sqrt{29}$

$\left| \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right| = \sqrt{29}$

$\therefore (\lambda-1)^2 + (\lambda+1)^2 + (2\lambda+3)^2 = 29$ ✓

$\lambda^2 - 2\lambda + 1 + \lambda^2 + 2\lambda + 1 + 4\lambda^2 + 12\lambda + 9 = 29$

$6\lambda^2 + 12\lambda - 18 = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda+3)(\lambda-1) = 0$ ✓

$\therefore \lambda = -3$ or $\lambda = 1$

when $\lambda = -3$: $P = (1-3, 0-3, 3-6) = (-2, -3, -3)$

When $\lambda = 1$: $Q = (1+1, 0+1, 3+2) = (2, 1, 5)$ ✓

* Points should not be written as column vectors.

Question 14 b)

iii) $PQ = \sqrt{(-2-2)^2 + (-3-1)^2 + (-3-5)^2}$

$= \sqrt{16+16+64}$

$= \sqrt{96}$

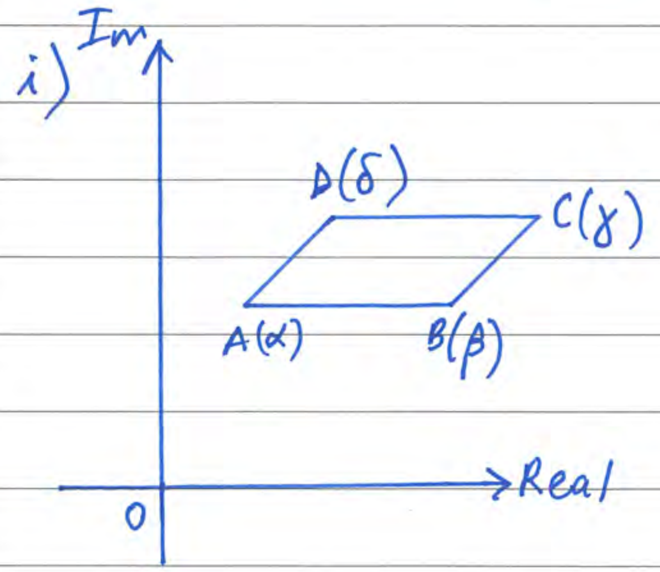
$= 2 \times \sqrt{24}$

$\neq 2 \times \sqrt{29}$ (where $\sqrt{29}$ is the radius of sphere S).

$\therefore$ PQ is NOT a diameter of the sphere.

Question 14 c) 2 + 2

i)



$\overrightarrow{BA} = \alpha - \beta \quad \overrightarrow{AB} = \beta - \alpha$

$\overrightarrow{CD} = \delta - \gamma \quad \overrightarrow{DC} = \gamma - \delta$

$\alpha = \overrightarrow{OA},\ \beta = \overrightarrow{OB},\ \gamma = \overrightarrow{OC},\ \delta = \overrightarrow{OD}$

Given: $\alpha + \gamma = \beta + \delta$

Show: ABCD is a parallelogram.

Solution: (alternatives considered).

$$\alpha + \gamma = \beta + \delta \quad \text{(given)}$$

$$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$$

$$\therefore \quad \overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

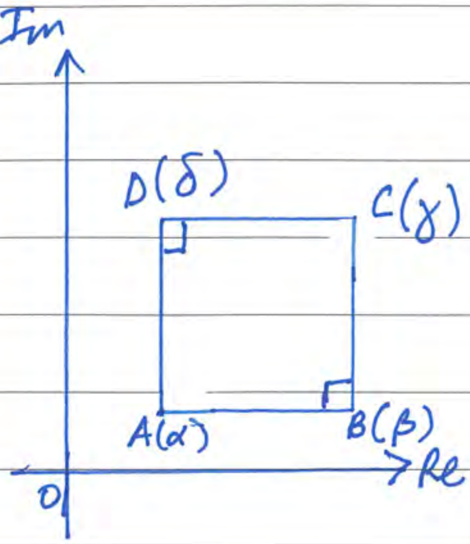
That is: $\overrightarrow{DC} = \overrightarrow{AB}$ ✓

* manipulation of the given statement

Hence, ABCD is a parallelogram given that one pair of opposite sides are both equal in length and parallel ✓ * Justification/reason.

Question 14 c) .

ii)



Given: ABCD is a square. vertices in anti-clockwise order

Show: $\gamma + i\alpha = \beta + i\beta$

Solution (1)

$\vec{BA} = i \times \vec{BC}$ $(AB \perp BC)$

$\alpha - \beta = i(\gamma - \beta)$ ✓

$i(\alpha - \beta) = -1(\gamma - \beta)$

$i\alpha - i\beta = -\gamma + \beta$ ✓

$\therefore \gamma + i\alpha = \beta + i\beta$ (as required).

* Marks were deducted if logic was not obvious.

OR Solution (2)

$\vec{AB} \times i = \vec{AD}$ $(AB \perp AD)$

$(\beta - \alpha)i = \delta - \alpha$ ✓

since $\vec{AD} \parallel \vec{BC}$

then $\delta - \alpha = \gamma - \beta$

$\therefore (\beta - \alpha)i = \gamma - \beta$

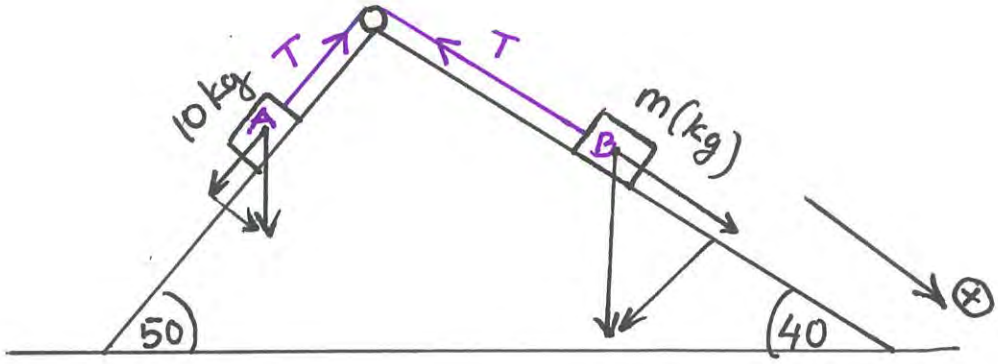
$i\beta - i\alpha = \gamma - \beta$ ✓

$\therefore \gamma + i\alpha = \beta + i\beta$ (as required).

* This part of the question needs revision by most students.

Q15

a)



$\boxed{1M}$

i) The two objects moving down the right hand side at a constant velocity, thus the acceleration of each body is zero. The acceleration of the system is zero $\underset{\sim}{a} = 0$. The net force $\underset{\sim}{F} = m\underset{\sim}{a}$

$$\underset{\sim}{F} = m \times 0 = 0$$

To get 1M, the terms "constant vel" OR "zero acceleration" must be used.

ii)

Forces on A (10 kg) object

$$T = 10g \sin 50^\circ \quad (1)$$

Forces on B (m kg) object

$$T = mg \sin 40^\circ \quad (2)$$

Solve (1) and (2)

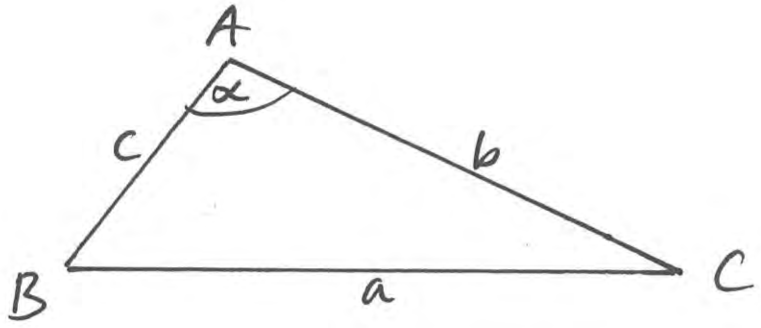
$$mg \sin 40^\circ = 10g \sin 50^\circ$$

$$m = \frac{10 \sin 50^\circ}{\sin 40} = \boxed{11.9 \text{ kg}}$$

Q15

b)

i)



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$ → This step only 1M can not be awarded.

$2bc$ is a rational as $r.s \in \mathbb{Q}$

$b^2+c^2-a^2$ is a rational as $r+s \in \mathbb{Q}$
$r-s \in \mathbb{Q}$

$\frac{b^2+c^2-a^2}{2bc} \in \mathbb{Q}$ as $\frac{r}{s} \in \mathbb{Q}$

OR $\cos\alpha$ is a rational $\therefore \cos\alpha \in \mathbb{Q}$.

ii) $(\cos\alpha + i\sin\alpha)^5 = \cos 5\alpha + i\sin 5\alpha$ ✓ ①

$(\cos\alpha + i\sin\alpha)^5 = \cos^5\alpha + 5\cos^4\alpha\, i\sin\alpha + 10\cos^3\alpha\, i^2\sin^2\alpha +$
$10\cos^2\alpha . i^3\sin^3\alpha + 5\cos\alpha . i^4\sin^4\alpha + i^5\sin^5\alpha$

$= \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha + (5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha)i$

② ✓

Equating real parts of ① and ②

$\cos 5\alpha = \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha$

$\cos 5\alpha = \cos^5\alpha - 10\cos^3\alpha(1-\cos^2\alpha) + 5\cos\alpha[1-\cos^2\alpha]^2$

$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$ ✓

as $\cos\alpha$ is a rational in part ⓘ

$\therefore 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$ is a rational

OR $\cos 5\alpha$ is a rational.

$\boxed{Q15}$ c/i) $\underset{\sim}{r_1} = \lambda \begin{bmatrix} \cos\phi + \sqrt{3} \\ \sqrt{2}\sin\phi \\ \cos\phi - \sqrt{3} \end{bmatrix} \quad : L_1$

$$\underset{\sim}{r_2} = \mu \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad : L_2$$

$$\underset{\sim}{r_1} \cdot \underset{\sim}{r_2} = |\underset{\sim}{r_1}| \cdot |\underset{\sim}{r_2}| \cdot \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{r_1} \cdot \underset{\sim}{r_2}}{|\underset{\sim}{r_1}| \cdot |\underset{\sim}{r_2}|} = \frac{\cos\phi + \sqrt{3} + 0 + \sqrt{3} - \cos\phi}{|\underset{\sim}{r_1}| \cdot |\underset{\sim}{r_2}|}$$

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{(\cos\phi + \sqrt{3})^2 + (\sqrt{2}\sin\phi)^2 + (\cos\phi - \sqrt{3})^2} \times \sqrt{1^2 + 0^2 + (-1)^2}}$$

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{\cos^2\phi + 2\sqrt{3}\cos\phi + 3 + 2\sin^2\phi + \cos^2\phi - 2\sqrt{3}\cos\phi + 3} \times \sqrt{2}}$$

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{2(\sin^2\phi + \cos^2\phi) + 6} \times \sqrt{2}}$$

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{8} \cdot \sqrt{2}} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$\cos\theta$ is independent of ϕ.

OR $\theta = \frac{\pi}{6}$. The acute angle between L_1 and L_2 is independent of ϕ.

Q15

c(ii) $l_2 : \underset{\sim}{r} = \mu \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ -\mu \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

subs into the plane

$$x - z = 4\sqrt{3}$$
$$\mu - -\mu = 4\sqrt{3}$$
$$2\mu = 4\sqrt{3}$$
$$\mu = 2\sqrt{3}$$

$\therefore$ The coordinate of C : $C(x, y, z)$

$C(2\sqrt{3}, 0, -2\sqrt{3})$ ✓

c(iii) $l_1 : x = \lambda(\cos\phi + \sqrt{3}), y = \sqrt{2}\lambda\sin\phi$

$z = \lambda(\cos\phi - \sqrt{3})$

Subs into the plane: $x - z = 4\sqrt{3}$

$$\lambda(\cos\phi + \sqrt{3}) - \lambda(\cos\phi - \sqrt{3}) = 4\sqrt{3} \checkmark$$
$$\lambda . 2\sqrt{3} = 4\sqrt{3}$$
$$\lambda = 2 \checkmark$$

$P(2(\cos\phi + \sqrt{3}), 2\sqrt{2}\sin\phi, 2(\cos\phi - \sqrt{3}))$

$PC = \sqrt{(2\cos\phi)^2 + (2\sqrt{2}\sin\phi)^2 + (2\cos\phi)^2} = \sqrt{8}$ ✓

$PC = 2\sqrt{2}$ $\therefore$ as ϕ varies, P traces out a circle of centre C and radius $2\sqrt{2}$.

AW 3 Marks for clearly showing the radius $= 2\sqrt{2}$ units.

Question 16

(a)

(i) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x < 0$ for $0 < x < \pi$, since $\sin x > 0$ for $0 < x < \pi$

$\therefore f(x)$ is concave down for $0 < x < \pi$.

Comment

Some students did not know they had to show that $f''(x) < 0$ to prove that $f(x)$ is concave down.

(ii) Let $f(x) = \sin x$, $x_1 = A$, $x_2 = B$, $x_3 = C$

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right)$$

$$= \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}$$

$$\therefore \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

Comment

Some students did not use the fact that $A + B + C = \pi$.

(b)

(i) $$\frac{x^{2n-1}}{\sqrt{1-x^2}} - \frac{x^{2n+1}}{\sqrt{1-x^2}} = \frac{x^{2n-1}(1-x^2)}{\sqrt{1-x^2}}$$

$$= x^{2n-1}\sqrt{1-x^2}$$

(ii) Integrate both sides of (i) :

$$\int_0^1 \frac{x^{2n-1}}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = \int_0^1 x^{2n-1}\sqrt{1-x^2}\,dx$$

$$I_{2n-1} - I_{2n+1} = \int_0^1 x^{2n-1}\sqrt{1-x^2}\,dx$$

$$u = \sqrt{1-x^2} \qquad v' = x^{2n-1}$$

$$u' = \frac{-x}{\sqrt{1-x^2}} \qquad v = \frac{x^{2n}}{2n}$$

$$I_{2n-1} - I_{2n+1} = \left[\frac{x^{2n}}{2n}\sqrt{1-x^2}\right]_0^1 + \frac{1}{2n}\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2n} I_{2n+1}$$

$$I_{2n-1} = \left(\frac{1}{2n} + 1\right) I_{2n+1}$$

$$= \frac{2n+1}{2n} I_{2n+1} \qquad \therefore I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$$

(iii) $$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3}$$

$$= \ldots$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-3} \cdot \ldots \cdot \frac{2}{3} I_1$$

$$I_1 = \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \left[-\sqrt{1-x^2}\right]_0^1$$

$$= 0-(-1)$$

$$= 1$$

$$\therefore I_{2n+1} = \frac{2n \times 2(n-1) \times \cdots \times 2(1)}{(2n+1)(2n-3)\cdots 3 \cdot 1}$$

$$= \frac{2^n n!}{1 \times 3 \times 5 \times \cdots \times (2n+1)}$$

Comment

students need to end with I_1 (in this case), and evaluate I_1 to be awarded full marks.

(iv) $$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \sum_{n=1}^{\infty} C_n \boxed{\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx}$$

$I_1 = 1$ $\quad$ I_{2n+1}

$$= 1 + \sum_{n=1}^{\infty} \frac{1\times3\times5\times\cdots\times(2n-1)}{(2n+1)2^n n!} \times \boxed{\frac{2^n \times n!}{1\times3\times5\times\cdots\times(2n+1)}}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

(v) $$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} + \sum_{n=1}^{\infty} C_n \frac{x^{2n+1}}{\sqrt{1-x^2}}$$

Integrating both sides w.r.t. x from $x=0$ to $x=1$:

$$\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \text{(from (iv))}$$

$$\left[\frac{(\sin^{-1}x)^2}{2}\right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\left(\frac{\pi}{2}\right)^2}{2}$$

$$\therefore\ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

(vi) $S = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)$ ✓

$= \frac{\pi^2}{8} + \left(\frac{1}{2^2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \cdots\right)$

$= \frac{\pi^2}{8} + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$

$= \frac{\pi^2}{8} + \frac{1}{4} S$

$\therefore \frac{3}{4} S = \frac{\pi^2}{8}$ $\quad\therefore S = \frac{\pi^2}{6}$ ✓